Problem 1. Let $f(x) = ax^2 + bx + c = a(x-h)^2 + k$ be be the unique quadratic function whose graph passes through the points $(-1,-4)$, $(0,-2)$, and $(1,6)$.

(a) Find a, b, c, h, k.

(b) Find the y-intercept, x-intercepts, and vertex.

(c) Sketch the graph.

Solution. Since the y-intercept is $(0,-2)$, we have $c = -2$. Thus $f(x) = ax^2 + bx - 2$. Now plug in $(-1,-4)$ and $(1,6)$ to get the system of two equations in two variables

$$\begin{aligned} a - b - 2 &= -4 \\ a + b - 2 &= 6 \end{aligned}$$

Adding these gives $2a - 4 = 2$, so $2a = 6$, so $a = 3$. Thus $b = 5$. Now $h = -\frac{b}{2a} = -\frac{5}{6}$, and $k = f(h) = -\frac{49}{12}$. Thus the vertex is $(h,k) = (-\frac{5}{6}, -\frac{49}{12})$. The x-intercepts are given by the quadratic formula as $(-2,0)$ and $(\frac{1}{3},0)$. □

Problem 2. Let $f(x) = a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0$ be the unique even quartic function (polynomial of degree four) with zeros at 2 and i and y-intercept $(0,-4)$.

(a) Find a_4, a_3, a_2, a_1, a_0.

(b) Sketch the graph of f.

(c) Find a point $(b, f(b))$ on the graph of f such that $f(b) \leq f(x)$ for all $x \in \mathbb{R}$.

Solution. Since f is even and 2 and i are a zeros, so are -2 and $-i$. Thus f is a constant times $(x-2)(x+2)(x-i)(x+i)$; but we see that the y-intercept of this function is $(0,-4)$, so it must be f. Thus

$$f(x) = x^4 - 3x^2 - 4.$$

To find a point $(b, f(b))$ on the graph where the minimum value of f occurs, we shift f up by a constant k, and find the $k > 4$ which produces two distinct solutions to $f(x) = 0$.

Consider the function $g(x) = f(x) + k = x^4 - 3x^2 + (k-4)$. Solving $g(x) = 0$ by the quadratic formula gives

$$x = \sqrt{\frac{3 \pm \sqrt{25 - 4k}}{2}}.$$

This has exactly two distinct solutions if $25 - 4k = 0$, and $k = \frac{25}{4}$. For $k > \frac{25}{4}$, there are no real solutions. This tells us that, if we shift the graph of f up by $\frac{25}{4}$, there will be two distinct real zeros, and if we shift it up by more that this, there will be no real zeros. Thus the minimum value for f is $-\frac{25}{4}$. This occurs at $b = \sqrt{\frac{3}{2}}$, and $f(b) = -\frac{25}{4}$. □

Problem 3. Let
$$f(x) = \frac{x^4 - 6x^2 + 5}{x^2 - 4}.$$

(a) Find all zeros and poles of f.

(b) Find the y-intercept, x-intercepts, vertical asymptotes, and polynomial asymptote of f.

(c) Sketch the graph of f.

Solution. We factor the numerator and denominator of the rational function an obtain
$$f(x) = \frac{(x^2-1)(x^2-5)}{x^2-4} = \frac{(x-1)(x+1)(x-\sqrt{5})(x+\sqrt{5})}{(x-2)(x+2)}.$$
Thus, the zeros of f are 1, -1, $\sqrt{5}$, $-\sqrt{5}$, and the poles are 2 and -2. Each of these has multiplicity 1, so the sign of f changes at each of them. The y-intercept is $(0, -\frac{5}{4})$ and the x-intercepts are $(1,0)$, $(-1,0)$, $(\sqrt{5},0)$, and $(-\sqrt{5},0)$. The vertical asymptotes are $x = 2$ and $x = -2$. The polynomial asymptote is the graph of $y = q(x)$, where $q(x)$ is the quotient of the numerator by the denominator. In this case, the parabolic asymptote is $y = x^2 - 2$.

Plotting all of this information produces the graph fairly accurately. □

Problem 4. Solve for x.

(a) $343^{(x-1)} = \frac{49^{(2x-2)}}{7^{(x-3)}}$

(b) $\log_{(x+5)}(17x + 13) = 2$

(c) $\log_{(x+1)}(3x + 5) + \log_{(x+1)} x = 3$

Solution. For **(a)**, we see that the common base is 7, and rewrite the equation as
$$7^{3x-3} = 7^{3x-1}.$$
This has no solution.

For **(b)**, we have
$$(x+5)^2 = 17x + 13 \quad \Rightarrow \quad x^2 - 7x + 12 = 0.$$
The solution are $x = 3$ or $x = 4$.

For **(c)**, we have $\log_{(x+1)}(x(3x+5)) = 2$, so
$$(x+1)^3 = x(5x+3) \quad \Rightarrow \quad x^3 - 2x + 1 = 0.$$
This has 1 as a solution; to find the other, use synthetic division to see that
$$x^3 - 2x + 1 = (x-1)(x^2 + x - 1).$$
By the quadratic formula, the other two zeros are $\frac{-1\pm\sqrt{5}}{2}$. The negative solution is less than negative one, and since negative bases for logarithm are disallowed, this is not a solution to the original equation. The solutions are $x = 1$ and $x = \frac{\sqrt{5}-1}{2}$. The latter solution is the reciprocal of the golden ratio. □

Problem 5. Compute the area of a regular fifteen sided polygon inscribed in a circle of radius one.

Solution. The polygon is the union of fifteen isosceles triangles with two sides of length 1 and acute angle equal to $\frac{360^\circ}{15} = 24^\circ$. The area is $A = \frac{1}{2}bh$, where $b = 1$ and $h = \sin 24^\circ$.

Thus the area is $A = \frac{15}{2}\sin 24^\circ$.

To compute $\sin 24^\circ$, we first compute the trigonometric function of 12° as

$$\begin{aligned}
\sin 12^\circ &= \sin(72^\circ - 60^\circ)\\
&= \sin 72^\circ \cos 60^\circ - \cos 72^\circ \sin 60^\circ\\
&= \frac{\sqrt{10+2\sqrt{5}}}{4}\cdot\frac{1}{2} - \frac{\sqrt{5}-1}{4}\cdot\frac{\sqrt{3}}{2}\\
&= \frac{\sqrt{10+2\sqrt{5}} - \sqrt{15} + \sqrt{3}}{8}.
\end{aligned}$$

and

$$\begin{aligned}
\cos 12^\circ &= \cos(72^\circ - 60^\circ)\\
&= \cos 72^\circ \cos 60^\circ + \sin 72^\circ \sin 60^\circ\\
&= \frac{\sqrt{5}-1}{4}\cdot\frac{1}{2} + \frac{\sqrt{10+2\sqrt{5}}}{4}\cdot\frac{\sqrt{3}}{2}\\
&= \frac{\sqrt{5} - 1 + \sqrt{30+6\sqrt{5}}}{8}.
\end{aligned}$$

Thus

$$\begin{aligned}
\sin 24^\circ &= 2\sin 12^\circ \cos 12^\circ\\
&= 2\Big(\frac{\sqrt{10+2\sqrt{5}} - \sqrt{15} + \sqrt{3}}{8}\Big)\Big(\frac{\sqrt{5} - 1 + \sqrt{30+6\sqrt{5}}}{8}\Big).
\end{aligned}$$

Simplifying this is asking for trouble, unless you are very careful. However, we have found an exact value for $\sin 24^\circ$. Using a calculator gives an approximate solution as $\sin 24^\circ \approx 0.41$, so $A \approx 3.05$. We believe this, because we know that the area of the enclosing circle is $\pi \approx 3.14$. □